Proof of the Maximum Principle for Two-Stage Fixed Endpoint Optimal Control[1]

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October 20, 2006

Abstract. Motivated by the growing use of two-stage optimal control in economic research, this paper offers a new and simple proof of the Maximum Principle for two-stage control problems with a fixed endpoint restriction on the state variable. In free endpoint problems the optimal control can be perturbed in any arbitrary manner in order to derive the necessary conditions, but in fixed endpoint problems the challenge is that perturbations to the optimal control must be feasible and therefore must guide the state variable from its initial position to the prescribed endpoint. Despite this complication, the proof of the Maximum Principle for two-stage problems in this paper is surprisingly straightforward and hinges on appropriately defining the transversality condition to ensure feasibility. This paper may be of value to economists looking for a short and intuitive discussion of the Maximum Principle for two-stage problems with a fixed endpoint.

Key Words: *Two-Stage Optimal Control; Fixed Endpoint; Maximum Principle*

JEL Classification: C61

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1 Introduction

A number of interesting problems in economics can be modeled as two-stage optimal control problems, wherein a switch in the objective function and/or state equation occurs at some point during the planning horizon. Some examples include: colonization (Kemp and Long 1977), patent expiration (Kemp and Long 1977), multigenerational environmental projects (Caliendo and Lyon 2005), technology adoption (Boucekkine et al. 2004), elimination of capital controls (Makris 2001), optimal investment with delivery lags (Tomiyama 1985), petroleum recovery (Amit 1986), and government price controls on resources (Kemp and Long 1977) to name just a few.

Some two-stage optimal control problems involve fixed endpoint restrictions on the state variable. One such example is a life-cycle permanent-income consumption model where the consumer cannot die with debt and has no bequest motive, but experiences discrete switches in the objective function over the life cycle due to changes in the number of children in the home or discrete switches in the state equation to reflect different sources of income over the life cycle (e.g., wage income during the working period and social security benefits during retirement). As noted in Bryson and Ho (1975, pp. 55-59), Kamien and Schwartz (1991, pp. 147-153), Long and Shimomura (2003), Caliendo and Pande (2005), and others, it is not a straightforward exercise to prove the necessary conditions for fixed endpoint problems. For example, in free endpoint problems the optimal control can be perturbed in any arbitrary manner in order to derive the necessary conditions. However, in fixed endpoint problems the challenge is that perturbations to the optimal control must be feasible and therefore must guide the state variable from its initial position to the prescribed endpoint. Bryson and Ho, Kamien and Schwartz, and Caliendo and Pande show how to derive necessary conditions using feasible perturbations, and Long and Shimomura side-step the feasibility problem in a clever way using the envelope theorem. However, the problem of choosing feasible perturbations becomes more complicated when there are switches in the control problem.

This paper offers a new and simple proof of the Maximum Principle for finite horizon two-stage control problems with a fixed endpoint restriction on the state variable. Despite the complications arising from the fixed endpoint aspect of the problem, the proof of the Maximum Principle in this paper is surprisingly straightforward and hinges on appropriately defining the transversality condition to ensure feasibility of modifications to the optimal control. This technique for ensuring feasibility comes from Caliendo and Pande's (2005) discussion of single-stage fixed endpoint problems, and the discussion is extended here to the two-stage problem. This paper may be of value to economists looking for a short and intuitive discussion of control problems of this type.

Finally, this is certainly not the first paper to modify the Maximum Principle to fit the two-stage framework. The first paper providing necessary conditions appears to be Kemp and Long (1977), followed by Tomiyama (1985), Amit (1986), and Makris (2001) and others. The contribution of the present paper is the straightforward *new proof* of the Maximum Principle that is based on a simple technique involving a new transversality condition.

2 The two-stage problem

The planning interval is fixed, finite, and is comprised of two stages ($i = 1, 2$). Each stage has a unique performance index and a unique state equation. Stage i starts at time a_i and ends at b_i. The control, u, is unconstrained and the state variable, x, is constrained only at the initial and terminal points in time. Other complications are avoided to focus on the fixed endpoint aspect of the two-stage problem. Thus, the two-stage problem that we consider is

$$\max : J = \sum_{i=1}^{2} \int_{a_i}^{b_i} f_i(t, x(t), u(t))dt \tag{1}$$

subject to

$$\begin{gathered} \frac{dx(t)}{dt} = g_i(t, x(t), u(t)), \qquad a_i \leq t \leq b_i, \qquad i = 1, 2 \\ x(a_1),\ x(b_2) \qquad \text{fixed} \\ a_2 = b_1 \end{gathered} \tag{2}$$

The first line in (2) provides the laws of motion for the state variable. The second line defines the initial and endpoint restrictions on the state variable; and, the last line is an assumption that ensures continuity of the two stages on the control interval (i.e., stages share common endpoints). The switch point, b_1, may be a choice variable or it may be treated as exogenous (we return to this point in Section 4).

3 Derivation of necessary conditions

Let $u^*(t),\ x^*(t)$ represent the optimal control and state variable, respectively. Now, we define two continuously differentiable multipliers, say $\lambda_i(t)$, for $i = 1, 2$. From (2) we have

$$\lambda_i(t) g_i(t, x(t), u(t)) - \lambda_i(t) \frac{dx(t)}{dt} = 0, \qquad a_i \leq t \leq b_i, \qquad i = 1, 2$$

Thus, J can be rewritten as

$$J = \sum_{i=1}^{2} \int_{a_i}^{b_i} \left\{ f_i(t) + \lambda_i(t) g_i(t) - \lambda_i(t) \frac{dx(t)}{dt} \right\} dt \tag{3}$$

where $f_i(t) \equiv f_i(t, x(t), u(t))$ and $g_i(t) \equiv g_i(t, x(t), u(t))$ to save space. Also, except when it is important to keep track of the time argument we will omit the "t" from functions $x,\ u,\ f,\ g,\ \lambda$.

Using integration by parts

$$\int_{a_i}^{b_i} \lambda_i \frac{dx}{dt} dt = \lambda_i(b_i) x(b_i) - \lambda_i(a_i) x(a_i) - \int_{a_i}^{b_i} \frac{d\lambda_i}{dt} x\ dt, \qquad i = 1, 2$$

Thus, (3) can be written

$$J = \sum_{i=1}^{2} \left\{ \int_{a_i}^{b_i} \left\{ f_i + \lambda_i g_i + \frac{d\lambda_i}{dt} x \right\} dt - \lambda_i(b_i) x(b_i) + \lambda_i(a_i) x(a_i) \right\} \tag{4}$$

The first variation of J is

$$\delta J = \sum_{i=1}^{2} \int_{a_i}^{b_i} \left[\left\{ \frac{\partial f_i}{\partial x} + \lambda_i \frac{\partial g_i}{\partial x} + \frac{d\lambda_i}{dt} \right\} \delta x + \left\{ \frac{\partial f_i}{\partial u} + \lambda_i \frac{\partial g_i}{\partial u} \right\} \delta u \right] dt - \sum_{i=1}^{2} \left\{ \lambda_i(b_i) \delta x(b_i) - \lambda_i(a_i) \delta x(a_i) \right\} \tag{5}$$

where $\delta u(t)$ is the modification to the optimal control and $\delta x(t)$ is the corresponding modification to the optimal state path. The partial derivatives of f and g in (5) are evaluated at the optimum $u^*(t),\ x^*(t)$. Thus, (5) gives the change in J resulting from a modification to the optimal control.

We can use the continuity assumption, i.e., $a_2 = b_1$, to write

$$\begin{aligned}\sum_{i=1}^{2}\{\lambda_i(b_i)\delta x(b_i) - \lambda_i(a_i)\delta x(a_i)\} &= \lambda_1(b_1)\delta x(b_1) - \lambda_1(a_1)\delta x(a_1) + \lambda_2(b_2)\delta x(b_2) - \lambda_2(a_2)\delta x(a_2) \\ &= \lambda_1(b_1)\delta x(b_1) - \lambda_1(a_1)\delta x(a_1) + \lambda_2(b_2)\delta x(b_2) - \lambda_2(b_1)\delta x(b_1)\end{aligned}$$

By definition, the modified state path starts at the same point as the optimal state path, thus $\delta x(a_1) = 0$ and we can substitute into (5)

$$\delta J = \sum_{i=1}^{2}\int_{a_i}^{b_i}\left[\left\{\frac{\partial f_i}{\partial x} + \lambda_i\frac{\partial g_i}{\partial x} + \frac{d\lambda_i}{dt}\right\}\delta x + \left\{\frac{\partial f_i}{\partial u} + \lambda_i\frac{\partial g_i}{\partial u}\right\}\delta u\right]dt - \left[\lambda_1(b_1) - \lambda_2(b_1)\right]\delta x(b_1) - \lambda_2(b_2)\delta x(b_2) \tag{6}$$

The multiplier functions are defined to obey

$$\frac{d\lambda_i(t)}{dt} = -\left[\frac{\partial f_i}{\partial x} + \lambda_i\frac{\partial g_i}{\partial x}\right], \qquad a_i \le t \le b_i, \qquad i = 1, 2 \tag{7a}$$

$$\lambda_1(b_1) = \lambda_2(b_1) \tag{7b}$$

Equation (7a) gives two distinct laws of motion for the multipliers, one for each separate stage, and (7b) is the traditional continuity condition which links the multipliers together at the switch point. We choose the modification

$$\delta u(t) = \frac{\partial f_i}{\partial u} + \lambda_i\frac{\partial g_i}{\partial u}, \qquad a_i \le t \le b_i, \qquad i = 1, 2 \tag{8}$$

Thus, from (7a), (7b), and (8), we can rewrite (6) as

$$\delta J = \sum_{i=1}^{2}\int_{a_i}^{b_i}\left\{\frac{\partial f_i}{\partial u} + \lambda_i\frac{\partial g_i}{\partial u}\right\}^2 dt - \lambda_2(b_2)\delta x(b_2)$$

Now, we assume that we can choose $\lambda_2(b_2)$ to ensure the chosen modification is feasible, i.e., we assume that $\lambda_2(b_2)$ can be selected so that the modified control drives the state from its initial position to the fixed endpoint and therefore renders $\delta x(b_2) = 0$. Thus, δJ becomes

$$\delta J = \sum_{i=1}^{2}\int_{a_i}^{b_i}\left\{\frac{\partial f_i}{\partial u} + \lambda_i\frac{\partial g_i}{\partial u}\right\}^2 dt \ge 0 \tag{9}$$

Thus, the chosen modification outperforms the optimal control unless

$$\frac{\partial f_i}{\partial u} + \lambda_i\frac{\partial g_i}{\partial u} = 0, \qquad a_i \le t \le b_i, \qquad i = 1, 2 \tag{10}$$

In sum, given (7a) and (7b), (10) is necessary for optimality.

Now, we must verify the earlier claim that the transversality condition $\lambda_2(b_2)$ can in fact be defined to ensure the feasibility of the modified control. We follow Caliendo and Pande (2005)—which in turn closely follows Kamien and Schwartz (1991, pp.147-153) and Bryson and Ho (1975, pp. 55-59)—but we modify their approach to fit the two stage problem.

Let $m_i(t)$ be a continuously differentiable multiplier, for $i = 1, 2$. From (2) we have

$$m_i(t)g_i(t, x(t), u(t)) - m_i(t)\frac{dx(t)}{dt} = 0, \qquad a_i \le t \le b_i, \qquad i = 1, 2 \tag{11}$$

Integrate from a_i to b_i to obtain

$$\int_{a_i}^{b_i} \left[m_i g_i - m_i \frac{dx}{dt} \right] dt = 0, \qquad i = 1, 2$$

and then sum across stages

$$\sum_{i=1}^{2} \int_{a_i}^{b_i} \left[m_i g_i - m_i \frac{dx}{dt} \right] dt = 0 \tag{12}$$

Using integration by parts, (12) can be written as

$$\sum_{i=1}^{2} \left\{ \int_{a_i}^{b_i} \left[m_i g_i + \frac{dm_i}{dt} x \right] dt - m_i(b_i)x(b_i) + m_i(a_i)x(a_i) \right\} = 0 \tag{13}$$

The first variation of (13) is

$$\sum_{i=1}^{2} \int_{a_i}^{b_i} \left[m_i \frac{\partial g_i}{\partial u} \delta u + \left(m_i \frac{\partial g_i}{\partial x} + \frac{dm_i}{dt} \right) \delta x \right] dt + \sum_{i=1}^{2} \{ m_i(a_i)\delta x(a_i) - m_i(b_i)\delta x(b_i) \} = 0 \tag{14}$$

where the partials of g are evaluated at $u^*(t),\ x^*(t)$. Now, we define $m_i(t)$ to be a solution to

$$\frac{dm_i}{dt} = -m_i \frac{\partial g_i}{\partial x}, \qquad a_i \le t \le b_i, \qquad i = 1, 2$$

Thus, (14) reduces to

$$\sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \delta u \ dt + \sum_{i=1}^{2} \{ m_i(a_i)\delta x(a_i) - m_i(b_i)\delta x(b_i) \} = 0 \tag{15}$$

which can be rewritten as

$$\sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \delta u \ dt + m_1(a_1)\delta x(a_1) - m_1(b_1)\delta x(b_1) + m_2(a_2)\delta x(a_2) - m_2(b_2)\delta x(b_2) = 0 \tag{16}$$

Now, using the continuity assumptions $a_2 = b_1$ and $m_1(b_1) = m_2(b_1)$, the assumption $m_2(b_2) = 1$, and the assumption $\delta x(a_1) = 0$, we can write (16) as

$$\sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \delta u \ dt = \delta x(b_2) \tag{17}$$

The key here is to show that $\lambda_2(b_2)$ can be chosen to ensure that the modification from (8) renders $\delta x(b_2) = 0$. To do this, solve for the general solution to (7a)

$$\lambda_i(t) = \left\{ C - \int^{t} \frac{\partial f_i(s)}{\partial x} \exp\left[\int^{s} \frac{\partial g_i(v)}{\partial x} dv \right] ds \right\} \exp\left[- \int^{t} \frac{\partial g_i(s)}{\partial x} ds \right] \tag{18}$$

where C is an arbitrary constant. Evaluate (18) at $t = b_i$ and solve for C

$$C = \lambda_i(b_i) \exp\left[\int^{b_i} \frac{\partial g_i(s)}{\partial x} ds \right] + \int^{b_i} \frac{\partial f_i(s)}{\partial x} \exp\left[\int^{s} \frac{\partial g_i(v)}{\partial x} dv \right] ds \tag{19}$$

Substitute (19) into (18) to obtain

$$\lambda_i(t) = \lambda_i(b_i) \exp\left[\int_{t}^{b_i} \frac{\partial g_i(s)}{\partial x} ds \right] + \int_{t}^{b_i} \frac{\partial f_i(s)}{\partial x} \exp\left[\int_{t}^{s} \frac{\partial g_i(v)}{\partial x} dv \right] ds \tag{20}$$

To save space, define

$$\gamma_i(t) \equiv \exp\left[\int_{t}^{b_i} \frac{\partial g_i(s)}{\partial x} ds \right]$$

$$\sigma_i(t) \equiv \int_{t}^{b_i} \frac{\partial f_i(s)}{\partial x} \exp\left[\int_{t}^{s} \frac{\partial g_i(v)}{\partial x} dv \right] ds$$

and rewrite (20) as

$$\lambda_i(t) = \lambda_i(b_i)\gamma_i(t) + \sigma_i(t) \tag{21}$$

Thus, the chosen modification from (8) can be written

$$\delta u(t) = \frac{\partial f_i}{\partial u} + (\lambda_i(b_i)\gamma_i + \sigma_i) \frac{\partial g_i}{\partial u}, \qquad a_i \le t \le b_i, \qquad i = 1, 2 \tag{22}$$

Now substitute (22) into (17)

$$\delta x(b_2) = \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \left[\frac{\partial f_i}{\partial u} + (\lambda_i(b_i)\gamma_i + \sigma_i) \frac{\partial g_i}{\partial u} \right] dt \tag{23}$$

which can be rewritten as

$$\begin{aligned} \delta x(b_2) &= \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \frac{\partial f_i}{\partial u}\, dt + \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \left(\frac{\partial g_i}{\partial u} \right)^2 \lambda_i(b_i)\gamma_i\, dt + \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \left(\frac{\partial g_i}{\partial u} \right)^2 \sigma_i\, dt \qquad (24) \\ &= \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \frac{\partial f_i}{\partial u}\, dt + \int_{a_1}^{b_1} m_1 \left(\frac{\partial g_1}{\partial u} \right)^2 \lambda_1(b_1)\gamma_1\, dt \\ &\quad + \int_{a_2}^{b_2} m_2 \left(\frac{\partial g_2}{\partial u} \right)^2 \lambda_2(b_2)\gamma_2\, dt + \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \left(\frac{\partial g_i}{\partial u} \right)^2 \sigma_i\, dt \end{aligned}$$

Now set $\delta x(b_2) = 0$ in (24) and solve for $\lambda_2(b_2)$ to obtain

$$\lambda_2(b_2) = \frac{\sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \frac{\partial g_i}{\partial u} \frac{\partial f_i}{\partial u} \, dt + \int_{a_1}^{b_1} m_1 \left(\frac{\partial g_1}{\partial u} \right)^2 \lambda_1(b_1)\gamma_1 \, dt + \sum_{i=1}^{2} \int_{a_i}^{b_i} m_i \left(\frac{\partial g_i}{\partial u} \right)^2 \sigma_i \, dt}{-\int_{a_2}^{b_2} m_2 \left(\frac{\partial g_2}{\partial u} \right)^2 \gamma_2 \, dt} \tag{25}$$

That is, defining the transversality condition $\lambda_2(b_2)$ as in (25) guarantees the modification (8) is feasible and renders $\delta x(b_2) = 0$.

Equation (25) is a fairly complex transversality condition. Fortunately, as in Caliendo and Pande (2005), this transversality condition holds automatically at the optimum and can be ignored in the statement of the necessary conditions. We prove this point in the appendix.

4 Concluding Remarks

Remark 1 *As some readers may have noticed, (25) contains $\lambda_1(b_1)$, which is a function of $\lambda_2(b_2)$ due to the continuity assumption on the multiplier. Thus, (25) is not an explicit transversality condition, but rather it implicitly defines $\lambda_2(b_2)$ as a function of itself. This issue can be rectified as follows. Evaluate (21) at $t = b_1$ and $i = 2$*

$$\lambda_2(b_1) = \lambda_2(b_2)\gamma_2(b_1) + \sigma_2(b_1) \tag{26}$$

Now impose continuity $\lambda_1(b_1) = \lambda_2(b_1)$

$$\lambda_1(b_1) = \lambda_2(b_2)\gamma_2(b_1) + \sigma_2(b_1) \tag{27}$$

Equation (27) can be substituted into (25) to uniquely define $\lambda_2(b_2)$.[3]

Remark 2 *Section 3 provides necessary conditions for optimality for any exogenous switch time. If the optimal switch time is an interior solution (lies between the endpoints of the planning interval), it is the solution to $f_1(b_1) + \lambda_1(b_1)g_1(b_1) = f_2(b_1) + \lambda_2(b_1)g_2(b_1)$. This is a standard "matching" condition and is a direct result of maximizing J from (3) with respect to b_1. For corner solutions see Tomiyama (1985) and Amit (1986).*

Remark 3 *The proof can easily be generalized to accommodate any finite number of exogenous switches. The only change will be the presence of additional continuity conditions on the multiplier at all the switch points.*

Remark 4 *In deriving the necessary conditions via the Bellman backward solution method (as in Kemp and Long 1977 and Tomiyama 1985), the continuity of the multiplier at the switch point is proven to be a necessary condition for optimal control (as long as there are no binding constraints on the state at the switch point—Kemp and Long), whereas in the present paper the continuity condition was assumed. Both approaches result in the same set of necessary conditions.*

[3] I thank Ngo Van Long for bringing this result to my attention.

5 Appendix

Here we prove that given (7a) and (10), (25) holds automatically on the optimal path. To initiate the proof, recall that the solution to (7a) is (21), i.e., the solution is

$$\lambda_i = \lambda_i(b_i)\gamma_i + \sigma_i \tag{A1}$$

Multiply (A1) by $m_i\left(\partial g_i/\partial u\right)^2$

$$m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i = m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i(b_i)\gamma_i + m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \sigma_i \tag{A2}$$

Integrate (A2)

$$\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i\ dt = \int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i(b_i)\gamma_i\ dt + \int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \sigma_i\ dt \tag{A3}$$

Sum over all i

$$\sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i\ dt = \sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i(b_i)\gamma_i\ dt + \sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \sigma_i\ dt \tag{A4}$$

From (10) we have $\lambda_i = -\dfrac{\partial f_i}{\partial u}/\dfrac{\partial g_i}{\partial u}$. Insert this into (A4)

$$-\sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \left[\frac{\partial f_i}{\partial u}/\frac{\partial g_i}{\partial u}\right]\ dt = \sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \lambda_i(b_i)\gamma_i\ dt + \sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \sigma_i\ dt \tag{A5}$$

Rewrite as

$$\begin{aligned} -\sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\frac{\partial g_i}{\partial u}\frac{\partial f_i}{\partial u}\ dt \quad &= \int_{a_1}^{b_1} m_1\left(\frac{\partial g_1}{\partial u}\right)^2 \lambda_1(b_1)\gamma_1\ dt + \int_{a_2}^{b_2} m_2\left(\frac{\partial g_2}{\partial u}\right)^2 \lambda_2(b_2)\gamma_2\ dt \\ &+ \sum_{i=1}^{2}\int_{a_i}^{b_i} m_i\left(\frac{\partial g_i}{\partial u}\right)^2 \sigma_i\ dt \end{aligned} \tag{A6}$$

Upon solving (A6) for $\lambda_2(b_2)$, we are left with (25). **Q.E.D.**

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